Finite speed of quantum scrambling with long range interactions

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In a locally interacting many-body system, two isolated qubits, separated by a large distance r, become correlated and entangled with each other at a time $t \geq r/v$ [1]. This finite speed v of quantum information scrambling limits quantum information processing [2], thermalization [3] and even equilibrium correlations [4]. Yet most experimental systems contain long range power law interactions – qubits separated by r have potential energy $V(r) \propto r^{-\alpha}$. Examples include the long range Coulomb interactions in plasma ($\alpha = 1$) and dipolar interactions between spins ($\alpha = 3$). In one spatial dimension, we prove that the speed of quantum scrambling remains finite for sufficiently large α. This result parametrically improves previous bounds [4–7], compares favorably with recent numerical simulations [8, 9], and can be realized in quantum simulators with dipolar interactions [10, 11]. Our new mathematical methods lead to improved algorithms for classically simulating quantum systems [6, 12], and improve bounds on environmental decoherence in experimental quantum information processors.

Almost five decades ago, Lieb and Robinson proved that spatial locality implies the ballistic propagation of quantum information [1]. Intuitively defining a "scrambling time" $t_\mathrm{s}(r)$ by the time at which an initially isolated qubit can significantly entangle with another a distance r away, locality implies that $t_\mathrm{s}(r) \gtrsim r$. This result has deep implications in theoretical physics: emergent spacetime locality arising from microscopic quantum mechanics without manifest relativistic invariance may play a crucial role in understanding quantum gravity through the holographic correspondence [13]. Moreover, if quantum information can only propagate with a finite speed, a classical computer can efficiently approximate early time quantum dynamics [12], and a quantum information processor with short-range interactions cannot become entangled with an infinite environment arbitrarily quickly [14, 15], despite the exponentially large Hilbert space in many-body quantum systems.

While the Lieb-Robinson theorem is quite elegant, it is not useful for a typical quantum information processor. A qubit in an experimental device is usually a spin or atomic degree of freedom, or Josephson junction. Such objects generically interact with long range interactions, and until now, whether locality of quantum scrambling necessarily persists in the presence of long range interactions has remained unclear. In 2005, Hastings and Koma used the canonical Lieb-Robinson theorem to prove that when $\alpha > d$, $t_\mathrm{s}(r) \gtrsim \log r$ [4]; more recently, this bound has been improved for $\alpha > 2d$ to $t_\mathrm{s}(r) \gtrsim r^{(\alpha-2d)/(\alpha-d)}$ [6]. If such bounds were tight, then insulating a quantum processor from its environment would be absolutely crucial. Yet numerical simulations cast into doubt the tightness of these formal bounds: two groups have recently shown that $t_\mathrm{s} \gtrsim r$ in one dimensional models with $\alpha \gtrsim 1.8$ [8] or even $\alpha > 1$ [9], depending on microscopic details.

In this letter, we prove that $t_\mathrm{s}(r) \gtrsim r$ whenever $\alpha > 3$, in all one dimensional models with power law interactions, and that in frustrated models, $t_\mathrm{s}(r) \gtrsim r$ whenever $\alpha > 2$. Our dramatic improvement over existing results is made possible by new mathematics [16]: identities for unitary time evolution expanded as a sum over flexibly chosen equivalence classes of sequences of couplings.

Our work has clear physical consequences. Scrambling in dipolar spin chains is hardly faster than in a spin chain with nearest neighbor interactions; hence, it should be far more efficient to simulate numerically [6, 12]. Nor does decoherence seriously limit the quantum information processing capabilities of a nuclear spin chain, no matter how large the environment. Quantum thermalization nearly proceeds as if interactions were local, as in typical theoretical models of scrambling [17, 18].

Formal Statement of Theorem.— We now formally restate our theorem in a mathematically precise language. Firstly, we are interested in one dimensional quantum many-body systems, which we define by the Hilbert space

$$\mathcal{H} = \bigotimes_{i\in\mathbb{Z}} \mathcal{H}_i. \tag{1}$$

The i in the above equation corresponds to an integer label on every lattice site; we assume $\dim(\mathcal{H}_i) < \infty$, or that the quantum degree of freedom on every site has a finite number of states. Even though $\mathcal{H}$ is (uncountably) infinite, our bound on scrambling will reduce to a calculation on an effectively finite dimensional Hilbert space.

The set of Hermitian operators on $\mathcal{H}$ forms a real vector space $\mathcal{B}$. Let the Lie group $\mathrm{U}(\dim(\mathcal{H}_i))$ generators $\{T_i^a\}$ be our complete basis of Hermitian operators on $\mathcal{H}_i$; we denote the identity on $\mathcal{H}_i$ as T_i^0. Then the following vectors span $\mathcal{B}$: $|a_i) = \bigotimes_{i\in\mathbb{Z}} T_i^{a_i}$. Here and below, we can use a "bra-ket" notation with parentheses to emphasize that Hermitian operators on $\mathcal{H}$ are vectors in $\mathcal{B}$. Through out the main text, we define $\|A\|$ as the maximal eigenvalue of A, the conventional operator norm [1].

We consider 2-local Hamiltonians: i.e., those which may be expressed as a sum of terms which act on either a single site, or on two sites:

$$H = \sum_{i\in\mathbb{Z}} H_i + \sum_{i<j} H_{ij}. \tag{2}$$

We define the exponent α of long range interactions by demanding that

$$\|H_{ij}\| \le \frac{h}{|i-j|^\alpha}. \tag{3}$$

Intuitively, H is frustrated if the interaction energy between two domains of L sites, separated by a distance $\ge L$, must decay as $\mathrm{O}(L^{1-\alpha})$ (if all H_{ij} are mutually commuting, then this energy could scale as $\mathrm{O}(L^{2-\alpha})$). A precise definition of frustration is contained in the supplementary material. A typical realization of a random ensemble of Hamiltonians will be frustrated. As noted previously, we can obtain stronger bounds for frustrated systems.

Finally, we use A_i and B_j to denote single site (1-local) operators acting on Hilbert spaces $\mathcal{H}_{i,j}$ respectively. We define the scrambling time $t^\delta_{\mathrm{s}}(r)$ to be the largest time such that for any $i,j\in\mathbb{Z}$:

$$\sup_{A_i,B_j} \frac{\|[A_i(s),B_j]\|}{\|A_i\|\|B_j\|} < \delta, \text{ for } 0<|s|<t^\delta_{\mathrm{s},a}(|i-j|). \tag{4}$$

This commutator norm constrains how a single-site operator ($c_{a_i}(0)=0$ if $i\neq 1$ and $a_i>0$) evolves into a more generic (sum of) tensor products of many such operators. In more physical terms, scrambling is the process by which a simple and local perturbation at time $t=0$ evolves into a highly complicated and non-local effect at later time t. In our paper we simply restrict ourself to definition (4), which bounds the growth in observable correlation functions, and the generation of entanglement between distant qubits [14, 15]. We are now ready to state our main result:

Theorem 1. *Define the parameter α' as follows:*

$$\alpha' = \begin{cases} \alpha & H \text{ is frustrated} \\ \alpha-1 & \text{otherwise} \end{cases}. \tag{5}$$

For every $0<\delta<2$, there exists a constant $0<K_\alpha<\infty$ for which

$$t^\delta_{\mathrm{s},a}(r) \ge K_\alpha \times \begin{cases} r^{\alpha'-1} & 1<\alpha'<2 \\ r(\log r)^{-2} & \alpha'=2 \\ r & \alpha'>2 \end{cases}. \tag{6}$$

Sketch of Proof.— We now outline the proof of Theorem 1; details are found in the supplement. For simplicity, we set $i=1$ and $j=r$ in (4), and $\dim(\mathcal{H}_j)=2$.

In the Heisenberg picture of quantum mechanics, operators evolve according to $\partial_t\mathcal{O} = \mathrm{i}[H,\mathcal{O}]$. Just like the Schrödinger equation, this is *linear*: we write $\partial_t|\mathcal{O}) = \mathcal{L}|\mathcal{O})$ where $\mathcal{L}$, commutation with Hamiltonian H, generates time translations on the space of operators. The time evolved operator $|\mathcal{O}(t)) = \mathrm{e}^{\mathcal{L}t}|\mathcal{O})$ is nothing more than a "rotated" operator of the same norm.

Any time evolving operator can be written as

$$|\mathcal{O}(t)) = \sum_{a_i} c_{a_i}(t)|a_i). \tag{7}$$

Thinking in terms of rotations, we define projection $\mathbb{P}$ onto the hyperplane Σ_r of $\mathcal{B}$ of all operators that act non-trivially on site r. This is a convenient object that bounds scrambling by the evolution of $|A_1)$ into Σ_r as a function of time: (see Figure 1)

$$\frac{\|[A_1(t),B_r]\|}{\|A_1\|\|B_r\|} < \|\mathbb{P}\mathrm{e}^{\mathcal{L}t}|A_1)\| \tag{8}$$

In the "canonical" form of the Lieb-Robinson theorem popularized by Hastings and Koma [4], one uses the triangle inequality: $\partial_t\|[A_1(t),B_r]\| \le \|[A_1(t),[H,B_r]]\|$. Yet most of the terms on the right hand side of this inequality sum do not contribute to $\|[A_1(t),B_r]\|$: they correspond to shifts in $A_1(t)$ that cannot grow $\|\mathbb{P}|A_1)\|$. We emphasize that this holds even though $\|A_1(t)\|$ is not the "length" of the vector $|A_1(t))$.

Instead, we write

$$\mathrm{e}^{\mathcal{L}t}|A_1) = \sum_{n=0}^\infty \frac{t^n}{n!}\sum_{X_1,\ldots,X_n}\mathcal{L}_{X_n}\cdots\mathcal{L}_{X_1}|A_1) \tag{9}$$

where X_i corresponds to a one or two-body coupling: e.g. $\mathcal{L}_{\sigma^x_1\sigma^x_2} = \mathrm{i}[\sigma^x_1\sigma^x_2,\cdot]$. As is known, $\mathcal{L}_{X_n}\cdots\mathcal{L}_{X_1}|A_1)$ is only non-zero if a subsequence of $\mathcal{L}$s form a path from 1 to r. Our main technical development is expanding $\mathbb{P}\mathrm{e}^{\mathcal{L}t}|A_1)$ in a controlled way: we classify all sequences with a path from 1 to r by a relatively small number of equivalence classes Γ. Generalizing the interacting picture, we obtain the following identity:

$$\mathbb{P}\sum_\Gamma \sigma(\Gamma)\int_0^t \mathrm{d}t_\ell \int_0^{t_\ell}\mathrm{d}t_{\ell-1}\cdots \mathrm{e}^{\mathcal{L}(t-t_\ell)}\mathcal{L}^\Gamma_\ell \mathrm{e}^{\tilde{\mathcal{L}}^\Gamma_\ell(t_\ell-t_{\ell-1})} \cdots \mathrm{e}^{\tilde{\mathcal{L}}^\Gamma_2(t_2-t_1)}\mathcal{L}^\Gamma_1\mathrm{e}^{\tilde{\mathcal{L}}^\Gamma_1 t_1}|A_1) = \mathbb{P}\mathrm{e}^{\mathcal{L}t}|A_1). \tag{10}$$

Here $\ell>0$, which depends on Γ, denotes the number of non-trivial steps $\mathcal{L}^\Gamma_j$, and $\sigma(\Gamma)=\pm1$ according to rules which we shortly state. Applying the triangle inequality to (10), and noting $\mathrm{e}^{\tilde{\mathcal{L}}^\Gamma_j t}$ is norm-preserving, which resums superfluous terms in the series expansion (9):

$$\frac{\|\mathbb{P}\mathrm{e}^{\mathcal{L}t}|A_1)\|}{2\|A_1\|} \le \sum_\Gamma \frac{t^\ell}{\ell!}\prod_{j=1}^\ell \|\mathcal{L}^\Gamma_j\|. \tag{11}$$

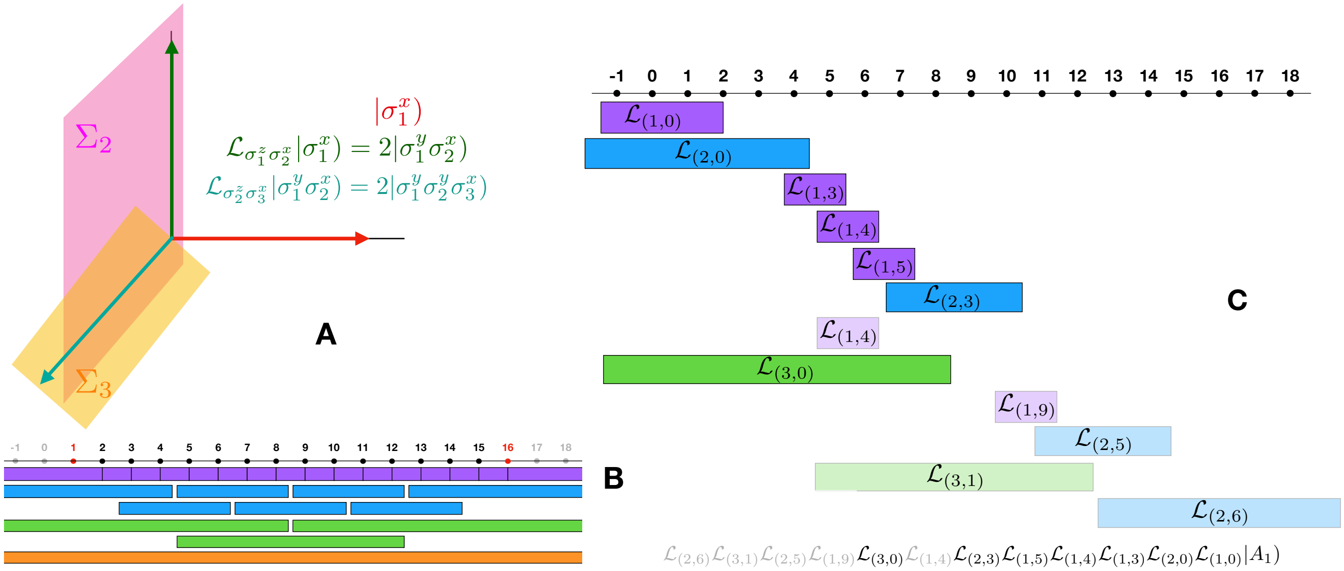

FIG. 1. **Methodology of proof.** (A) A growing operator $A_i(t)$ is interpreted as a sum of vectors in the vector space of all Hermitian operators. As t increases, $A_i(t)$ will increasingly point in the direction of large operators: products of many Paulis on different sites, which lie in the intersection of many hyperplanes Σ_j, corresponding to operators acting non-trivially on site j. (B) The space of all couplings $\mathcal{L}_{ij}$ $(i<j)$ can be broken up into the *scale* on which the coupling acts in a unique way. Intuitively, the scale q of a coupling is approximately $\lceil \log_2(j-i)\rceil$. Each scale is denoted with a different color: from large (red) to short (purple). In this example, we study $\|[A_1(t), B_{16}]\|$, and sites n obeying $n<1$ or $n>16$ are grouped in with these end sites when combining couplings. (C) Any sequence of $\mathcal{L}$ which grows $A_1(t)$ to the final site 16 must have a long sequence of couplings on at least one scale. For the particular sequence shown, there are three scales with sufficiently long sequences (no shorter than $16/\log 16) = 4$), and we bound the contribution of this sequence to $\|[A_1(t), B_{16}]\|$ by summing over the weight of all possible paths which contain the solid colored couplings (corresponding to $\mathcal{L}_j^\Gamma$) in a precise order. The lightly shaded couplings (corresponding to $\tilde{\mathcal{L}}_j^\Gamma$) do not contribute to (11).

where

$$\|\mathcal{L}_j^\Gamma\| := \sup_{\mathcal{O}} \frac{\|\mathcal{L}_j^\Gamma \mathcal{O}\|}{\|\mathcal{O}\|} \tag{12}$$

The choice of Γ is quite flexible. For this Hamiltonian, our construction is depicted in Figure 1. We start by regrouping all $\mathcal{L}_{mn} = \mathrm{i}[H_{mn},\cdot]$ by the scale q at which they act: roughly $q = \lfloor \log_2|m-n|\rfloor$. We write H as a sum of one dimensional Hamiltonians, each consisting of terms of a given scale. At scale $q=0$ $(q>0)$, these blocks form a one dimensional model of nearest (next nearest) neighbor interactions. Observe that any path from 1 to r must traverse forward a distance $\gtrsim r/\log r$ on *at least one of the* $\log_2 r$ *scales*. Since this criteria for a scale q is independent of any other scale q', we can invoke the inclusion-exclusion principle to enumerate all satisfying paths exactly once. The equivalence classes Γ are labeled by a non-empty subset of the integers $\{0,1,\ldots,\lfloor\log_2 r\rfloor\}$ corresponding to the scales on which a "long" path from 1 to r exists; $\sigma(\Gamma) = (-1)^{1+|\Gamma|}$ comes from inclusion-exclusion; $\mathcal{L}_j^\Gamma$ in (10) correspond to one block of couplings (as depicted in Figure 1); $\tilde{\mathcal{L}}_j^\Gamma$ prevent any coupling from modifying the right-moving sequence of length $2^{-q}r/\log_2 r$, for any q.

In fact, we can improve this argument in a few ways. (*1*) We demand that paths at scale q have length at least N_q, with N_q tuned so that the contribution of all scales q to (11) is comparable. (*2*) We demand that all "long" paths must increase the right-most site on which the operator acts. (*3*) We optimize $\|\mathcal{L}_j^\Gamma\|$ for different choices of operator norm, and obtain $\|\mathcal{L}_j^\Gamma\| \propto 2^{-q(\alpha'-1)}$. We then evaluate (11). Our results are summarized below.

When $\alpha'>2$, the dominant contribution to $\|\mathbb{P}\mathrm{e}^{\mathcal{L}t}|A_1)\|$ comes from short length scales: a large fraction of the path from 1 to r often occurs in *nearest neighbor* hops. Scrambling proceeds as if interactions were nearest neighbor alone. The operator $|A_1(t))$ is largely supported on lattice sites $x<vt$, where v is a finite speed of quantum scrambling.

When $\alpha'<2$, the dominant contribution to $\|\mathbb{P}\mathrm{e}^{\mathcal{L}t}|A_1)\|$ comes from few long hops across 1 to r. Counting the number of such long hops, we find $t_\mathrm{s}(r) = \mathrm{O}(r^{\alpha-1})$.

Lastly, if $\alpha'=2$, we find that all scales are equally important, which leads to $t_\mathrm{s}(r) = \mathrm{O}(r/\log^2 r)$.

Outlook.— We conclude the letter with a discussion of the implications of our theorem. Recall that our new mathematical methods led to dramatic improvements over existing literature, where the previous optimal bound on scrambling in one dimensional systems was $t_\mathrm{s}(r) \gtrsim r^{(\alpha-2)/(\alpha-1)}$ for $\alpha>2$ [6]. In fact, for any $\alpha>3$, the speed of quantum scrambling is finite: entanglement [14, 15] and quantum state transfer [2] proceed at a finite rate, and thermalization largely mimics that of a locally interacting system.

Our results are very similar to the numerical simulations of [8], where it was argued that a finite speed of scrambling arises for $\alpha \gtrsim 1.8$ in a model with time-dependent random Hamiltonian. However, in another model with fixed Hamiltonian [9], it was found that $\alpha \gtrsim 1$ marked the onset of the finite scrambling speed. We expect that (6) holds with $\alpha' = \alpha$ for all models, including those which are not (by our definition) frustrated. It would be interesting if this can be proved rigorously.

The techniques developed in this letter may generalize to other important problems in quantum information dynamics, including entanglement growth and quantum scrambling in finite temperature states. We also hope to generalize our main theorem to any spatial dimension d. Lastly, we have also used similar techniques to constrain models of holographic quantum gravity [16]. Given the recent explosion of interest in realizing analogue black holes in quantum simulators [19, 20], our methods will lead to sharp constraints on which experiments are capable of achieving this ambitious goal.

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SUPPLEMENTARY MATERIAL

The supplementary material to this letter contains the formal proof of Theorem 1, stated in the main text. The notation follows that introduced in the main text.

Proof of Theorem 1: Without loss of generality, we may take the starting vertice to be on the left $i < j$. Define

$$n_* = \lfloor \log_2 |j - i| \rfloor , \tag{S1}$$

define

$$R = 2^{n_*} \tag{S2}$$

and

$$i' = j - R + 1. \tag{S3}$$

This last equation is used to push the starting vertex i farther forward than it actually is, in order to simplify some equations that follow. We define the projection superoperators $\mathbb{P}_j$ as follows:

$$\mathbb{P}_1|\{a_i\}) = |\{a_i\}) \times \mathbb{I}(a_m = 0 \text{ if } m > i'), \tag{S4a}$$
$$\mathbb{P}_k|\{a_i\}) = |\{a_i\}) \times \mathbb{I}(a_{k+i'} \neq 0)\mathbb{I}(a_m = 0 \text{ if } m > k + i'), \tag{S4b}$$
$$\mathbb{P}_R|\{a_i\}) = |\{a_i\}) \times \mathbb{I}(a_m \neq 0 \text{ for some } m \geq j). \tag{S4c}$$

Here $\mathbb{I}(\cdots)$ denotes the indicator function, which returns 1 if its argument is true, and 0 if false. The first step is that we can replace the commutator norm $\|[A_i(t), B_j]\|$ of Lieb and Robinson by the length of the projection $\|\mathbb{P}_j|A_i(t))\|$:

Proposition 2. *If $\mathcal{O}$ is a Hermitian operator, and B_R is 1-local on site R,*

$$\sup_{B_R} \frac{\|[\mathcal{O}, B_R]\|}{\|\mathcal{O}\|\|B_R\|} \leq 2\frac{\|\mathbb{P}_R\mathcal{O}\|}{\|\mathcal{O}\|} \tag{S5}$$

Proof. Starting with $[\mathcal{O}, B_R] = [\mathbb{P}_R\mathcal{O}, B_R]$, by submultiplicativity and triangle inequality

$$\frac{\|[\mathcal{O}, B_R]\|}{\|B_R\|} = \frac{\|[\mathbb{P}_R\mathcal{O}, B_R]\|}{\|B_R\|} \leq 2\|\mathbb{P}_R\mathcal{O}\| \tag{S6}$$

□

Proposition 3. *$\mathbb{P}_R$ cannot arbitrarily grow the norm of an operator:*

$$\|\mathbb{P}_R\mathcal{O}\| \leq 2\|\mathcal{O}\| \tag{S7}$$

Proof. Let $\dim(H_R) = d$. Using the Lie algebra identity [16]

$$\mathbb{P}_R\mathcal{O} = -\frac{1}{2d^2}\sum_{a=1}^{d^2-1}[T^a_R, [T^a_R, \mathcal{O}]] \tag{S8}$$

where T^a_R are normalized such that $\mathrm{tr}((T^a_R)^2) = d$, and using submultiplicativity and triangle inequality, together with the fact that $\|T^a_R\| \leq \sqrt{d}$:

$$\|\mathbb{P}_R\mathcal{O}\| \leq 2\left(d - \frac{1}{d}\right)\|\mathcal{O}\| \leq 2d\|\mathcal{O}\|$$

□

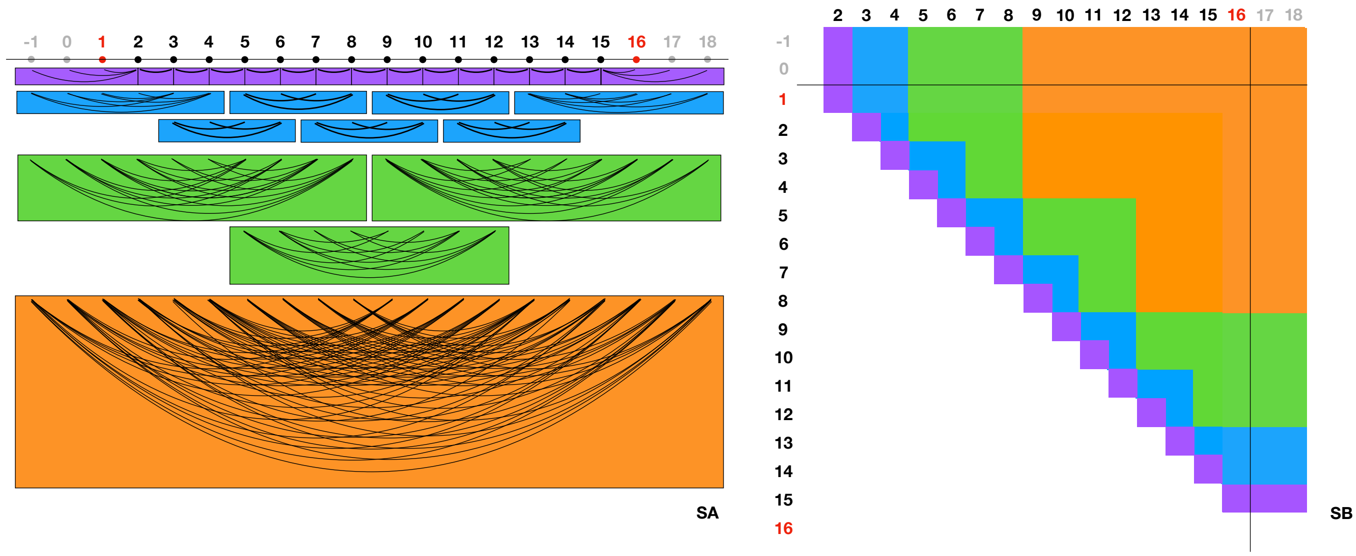

FIG. S1. **Decomposition of couplings into multiple scales.** (SA) Every single coupling $\mathcal{L}_{ij}$ is shown grouped into the blocks $\mathcal{L}_{(q,k)}$; (SB) a two dimensional representation of the same grouping. Different colors correspond to different values of q.

Note that when $d = 2$ in the above proof, the constant $2d$ in the final bound can be replaced with 2: the T_R^a are the Pauli matrices, which all have operator norm 1.

The next step of the proof, as sketched in the main text, is to organize the sequences of Liouvillians $\mathcal{L}_{X_n} \cdots \mathcal{L}_{X_1}$ in (9) by paths from i' to j on multiple different scales (fig: 1). Given two non-negative integers $q \geq 1$ and $k \geq 0$, we define the sets

$$\mathcal{Q}(1,k) := \{\{k+1, k+2\}\}, \tag{S9a}$$

$$\mathcal{Q}(q,k) := \{\{i,j\} : 1 \leq i < j \leq R, \quad 2^{q-1}k < i < j \leq 2^{q-1}(k+2)\} - \bigcup_{k' \geq 0, q' < q} \mathcal{Q}(q', k'), \quad (q > 1). \tag{S9b}$$

These sets contain all the couplings (at each scale) which can propagate information forward, and will be used to reduce the problem to a simpler calculation on a one dimensional line with nearest neighbor interactions. We reorganize the 2-local Liouvillians $\mathcal{L}_{ij}$ according to $\mathcal{Q}(q,k)$:

$$\mathcal{L} = \sum_{(q,k)} \mathcal{L}_{(q,k)} \tag{S10}$$

where

$$\mathcal{L}_{(q,k)} := \sum_{\{i,j\} \in \mathcal{Q}(q,k)} \widetilde{\mathcal{L}}_{ij}. \tag{S11}$$

and we define the shifted 2-local Liouvillians to take care for interaction with longer that $|i-j|$ (e.g. $L_{i-10,j+4}$):

$$\widetilde{\mathcal{L}}_{mn} := \begin{cases} \mathcal{L}_{i'+m-1, i'+n-1} & 1 < m < n < R \\ \sum_{k \leq i'} \mathcal{L}_{k, i'+n-1} & 1 = m < n < R \\ \sum_{k \geq j} \mathcal{L}_{i'+m-1, k} & 1 < m < n = R \\ \sum_{k \leq i, k' \geq j} \mathcal{L}_{k,k'} & 1 = m < n = R \end{cases}. \tag{S12}$$

See Figure S1 for a pictorial representation of the grouping of couplings.

One definition of a frustrated Hamiltonian is that there exists a constant K such that for all (q,k):

$$K \| H_{(q,k)} \| \leq \| H_{(q,k)} \|_2, \tag{S13}$$

with constant $0 < K < \infty$ independent of q.

Lemma 4. *The super-operator norm is bounded by*

$$\|\mathcal{L}_{(q,k)}\| \leq \frac{b}{2^{q(\alpha'-1)}} \tag{S14}$$

where

$$b := h \times \begin{cases} \dfrac{2^{2\alpha-\frac{1}{2}}}{(\alpha-1)K} & \text{frustrated model} \\ \dfrac{2^{\alpha+2}}{(\alpha-1)(\alpha-2)} & \text{any } H \end{cases} . \tag{S15}$$

Proof. Case 1: Frustrated models. Observe that

$$\begin{aligned} \|\mathcal{L}_{(q,k)}\| &= \sup_{\mathcal{O}} \frac{\|\mathcal{L}_{(q,k)}\mathcal{O}\|}{\|\mathcal{O}\|} \leq 2\|H_{(q,k)}\| = \frac{2}{K}\sqrt{\frac{\mathrm{tr}\left(H^2_{(q,k)}\right)}{\dim(\mathcal{H}_{(q,k)})}} \\ &\leq \frac{2}{K}\sqrt{\sum_{\{i,j\}\in\mathcal{Q}(q,k)} \|\widetilde{H}_{ij}\|^2} \\ &\leq \frac{2}{K}\sqrt{\sum_{i=2^q(k+\frac{1}{2})-1}^{-\infty}\sum_{j=2^q(k+1)}^{\infty}\frac{h^2}{|j-i|^{2\alpha}} + \sum_{i=2^q(k+1)-1}^{-\infty}\sum_{j=2^q(k+\frac{3}{2})}^{\infty}\frac{h^2}{|j-i|^{2\alpha}}} \\ &< \frac{2}{K}\sqrt{2}\sqrt{\sum_{i=2^q(k+\frac{1}{2})-1}^{-\infty}\sum_{j=2^q(k+1)}^{\infty}\frac{h^2}{|j-i|^{2\alpha}}} < \frac{2}{K}h\sqrt{2}\sqrt{\sum_{m,n=0}^{\infty}\frac{1}{|2^{q-1}+m+n|^{2\alpha}}} \\ &< \frac{2}{K}h\sqrt{2}\sqrt{\sum_{m,n=1}^{\infty}\frac{2^{2\alpha}}{|2^{q-1}+m+n|^{2\alpha}}} < \frac{2^{\alpha+\frac{3}{2}}h}{K}\sqrt{\int_0^{\infty}\mathrm{d}m\int_0^{\infty}\mathrm{d}n\frac{1}{(2^{q-1}+m+n)^{2\alpha}}} \\ &< \frac{2^{\alpha+\frac{3}{2}}h}{K}\sqrt{\int_0^{\infty}\mathrm{d}m\frac{1}{(2\alpha-1)(2^{q-1}+m)^{2\alpha-1}}} < \frac{2^{\alpha+\frac{3}{2}}h}{K\sqrt{(2\alpha-1)(2\alpha-2)2^{(q-1)(\alpha-1)}}} \end{aligned} \tag{S16, S17}$$

where in the first line, we used the triangle inequality and submultiplicativity ($\|AB\| \leq \|A\|\|B\|$); in the second we used the fact that the product of two non-trivial two-body operators acting on non-identical degrees of freedom must be traceless; in the third line we constrained all possible pairs $\{i,j\}$ in $\mathcal{Q}(q,k)$; in the fourth line we employed (3), and the remainder of inequalities are elementary manipulations.

Case 2: Any H. We simply use triangle inequality on $\|H_{(q,k)}\|$:

$$\begin{aligned} \|\mathcal{L}_{(q,k)}\| &\leq 2\|H_{(q,k)}\| \leq 2\sum_{i=2^q(k+\frac{1}{2})-1}^{-\infty}\sum_{j=2^q(k+1)}^{\infty}\frac{h}{|j-i|^{\alpha}} + \sum_{i=2^q(k+1)-1}^{-\infty}\sum_{j=2^q(k+\frac{3}{2})}^{\infty}\frac{h}{|j-i|^{\alpha}} \\ &< 2h\sum_{i=2^q(k+\frac{1}{2})-1}^{-\infty}\sum_{j=2^q(k+1)}^{\infty}\frac{1}{|i-j|^{\alpha}} < 4h\sum_{m,n=1}^{\infty}\frac{2^{\alpha}}{|2^{q-1}+m+n|^{\alpha}} < \frac{2^{\alpha+2}h}{(\alpha-1)(\alpha-2)2^{(q-1)(\alpha-2)}} \end{aligned} \tag{S18}$$

□

Let $\beta_i = (q,k)$ denote one of the sets of couplings at scale q described above. For convenience, when $\beta = (q,k)$, we will write $q(\beta) = q$ and $k(\beta) = k$. Let $(\beta_1, \ldots, \beta_n)$ denote an ordered sequence of Liouvillians $\mathcal{L}_{\beta_n}\cdots\mathcal{L}_{\beta_1}$.

Lemma 5. *Every non-vanishing sequence must satisfy*

$$k(\beta_1) = 0 \tag{S19a}$$

$$2^{q(\beta_m)-1}k(\beta_m) + 1 \leq \max_{1\leq m'<m}\left(2^{q(\beta_{m'})-1}(k(\beta_{m'})+2)\right). \tag{S19b}$$

This kind of sequence $\beta = (\beta_1, \ldots, \beta_n)$ is an instance of a broader notion called creeping [16] applied to this system.

Proof. This proof also follows [16] and is straightforward. $\mathcal{L}_{\beta_n}\cdots\mathcal{L}_{\beta_1}|A_1)\neq 0$ implies $\mathcal{L}_{\beta_1}$ overlaps with site A_1, which implies $k(\beta_1)=0$. Next, let $J := \max_{1\le m'<m}(2^{q(\beta_{m'})-1}(k(\beta_{m'})+2))$ and $|A_{m-1}) := \mathcal{L}_{\beta_{m-1}}\cdots\mathcal{L}_{\beta_1}|A_0)$. Following the prior logic, $\mathbb{P}_p|A_{m-1})=0$ if $p>J$. If $2^{q(\beta_m)-1}k(\beta_m)>J$, then $\mathcal{Q}(\beta_m)$ does not overlap with the farthest site$\{x\le J\}$, and hence $\mathcal{L}_{\beta_m}|A_{m-1})=0$. Therefore if $\mathcal{L}_{\beta_m}\neq 0$, the sequence must be creeping. □

We say that a sequence $\beta=(\beta_1,\ldots,\beta_n)$ is a *forward* sequence from j_1 to j_2 if for all $1\le m<n$, $2^{q(\beta_m)-1}(k(\beta_m)+2)<2^{q(\beta_{m+1})-1}(k(\beta_{m+1})+2)$, and if $2^{q(\beta_1)-1}k(\beta_1)=j_1$ and $2^{q(\beta_n)-1}(k(\beta_n)+2)=j_2$. As we will see in Lemma 6, every creeping sequence from 1 to R must have a sufficiently "long" forward subsequence, and these forward sequences will then play a crucial role in our proof. We define

$$N_q = \left\lceil \frac{1}{2}\frac{2^{-q(\alpha'-2)/2}}{\sum_{q'=1}^{n_*}2^{-q'(\alpha'-2)/2}}\frac{R}{2^q}\right\rceil \tag{S20}$$

to be the number of couplings at scale q which makes a sequence "long" – in our context, we chose N_q such that the long paths at each scale contributes to the commutator norm slowly and somewhat "equally" between all scales. (This will be proven towards the end of our proof of the theorem.) We say that a forward sequence from i' to j is a *long q-forward sequence* from 1 to R if (*1*) it contains a forward subsequence of length N_q, $\beta_q=(\beta_{i_1},\ldots,\beta_{i_{N_q}})$ with the same scale $q=q(\beta_{i_m})$ for all $1\le m\le N_q$, and (*2*) any forward subsequence β' remains forward if any element of β_q is added to the sequence β'. In simpler terms, this forward subsequence must correspond to a sequentially increasing sequence of couplings at scale q, each of which also can grow the operator to the right. As a matter of bookkeeping, we denote subsequence β' of β as $\beta'\subseteq\beta$ and define characteristic functions χ_q to indicate sequences with long q-subsequences:

$$\chi_q\mathcal{L}_{\beta_p}\cdots\mathcal{L}_{\beta_1} := \begin{cases}\mathcal{L}_{\beta_p}\cdots\mathcal{L}_{\beta_1} & \text{if there exists long } q\text{-forward subsequence } \beta'\subseteq\beta \\ 0 & \text{else}\end{cases}. \tag{S21}$$

Having proven the lemmas above, we now set the stage for the remainder of the proof. Let $\mathcal{S}$ denote the set of all creeping sequences which contain a forward subsequence from 1 to R,

$$\mathbb{P}_R e^{\mathcal{L}t}|A_1) = \mathbb{P}_R\sum_{p=0}^{\infty}\frac{t^p}{p!}\mathcal{L}^p|A_1) = \mathbb{P}_R\sum_{p=0}^{\infty}\frac{t^p}{p!}\sum_{\beta\in\mathcal{S}:|\beta|=p}\mathcal{L}_{\beta_p}\cdots\mathcal{L}_{\beta_1}|A_1). \tag{S22}$$

Naively bounding (S22) would lead to a lousy bound. The main idea is that we can repackage these terms using the iinclusion-excluison principle, where each group of term resums nicely. We exclude the paths without long q-forward sequences for any q: such paths vanish, as they cannot creep far enough to reach R, as shown by the following lemma:

Lemma 6. *If $\beta=(\beta_1,\ldots,\beta_n)$ is creeping and $\mathcal{L}_{\beta_n}\cdots\mathcal{L}_{\beta_1}|A_0)\neq 0$, then it has a long q-forward subsequence for at least one integer $0\le q\le n_*$.*

Proof. We proceed in two steps, first showing that we can always construct a (possibly empty) q-forward subsequence of any creeping $(\beta_1,\ldots,\beta_n)$, and secondly showing that at least one of the sequences must be large.

Firstly, we explicitly construct a q-forward subsequence $\beta^q\subseteq\beta$ as follows. Start with an empty sequence $\beta^q=()$; then read the sequence β in order. If an m at which $q(\beta_m)=q$ is found, and $(k(\beta_m)+2)2^{q(\beta_m)-1}>(k(\beta_{m'})+2)2^{q(\beta_{m'})-1}$ for any $m'<m$, set $\beta^q=(\beta_m)$. Afterwards, suppose that the current sequence β^q terminates with coupling β_{m_0} and that we have read β up to coupling m. If $q(\beta_m)=q$ and $(k(\beta_m)+2)2^{q(\beta_m)-1}>(k(\beta_{m'})+2)2^{q(\beta_{m'})-1}$ for all $m'<m$, replace $\beta^q\to(\beta^q,\beta_m)$. The final sequence β^q which we obtain is the output of this algorithm. By construction, this is a forward (sub)sequence made out of only q-scale couplings, so it is q-forward. The sequence β^q need not be creeping.

For a contradiction, suppose that none of the q-forward subsequences found above are long. Let $\hat{\beta}$ be the maximal forward subsequence of β; note that $\beta^1\cup\cdots\cup\beta^{n_*}=\beta$. If the sequence crept all the way beyond R, then trivially we have

$$R<\sum_{p=1}^{\ell(\hat{\beta})}2^{q(\hat{\beta}_p)}. \tag{S23}$$

By definition, every coupling that shows up in the forward sequence $\hat{\beta}$ must show up in a q-forward sequence for some q, so

$$\sum_{p=1}^{\ell(\hat{\beta})} 2^{q(\hat{\beta}_p)} < \sum_{q=1}^{n_*} 2^q \ell(\beta^q). \tag{S24}$$

Now, by assumption every q-forward subsequence β^q had $\ell(\beta^q) < N_q$, and we arrive at a contradiction:

$$R < \sum_{q=1}^{n_*} 2^q \ell(\beta^q) < \frac{1}{2} \frac{\sum_{q=1}^{n_*} 2^q \times \frac{R}{2^q} 2^{-q(\alpha-2)/2}}{\sum_{q'=1}^{n_*} 2^{-q'(\alpha-2)/2}} = \frac{R}{2}. \tag{S25}$$

□

The next step is to convert Lemma 6 into an explicit identity of the form (10).

Proposition 7.

$$\begin{aligned}
\mathbb{P}_R \sum_{p=0}^{\infty} \frac{t^p}{p!} \mathcal{L}^p |A_1) &= \left[1 - \prod_{q=1}^{n_*} (1-\chi_q)\right] \mathbb{P}_R \sum_{p=0}^{\infty} \frac{t^p}{p!} \mathcal{L}^p |A_1) \\
&= \left[\sum_q \chi_q + \sum_{q_1<q_2} \chi_{q_1}\chi_{q_2} + \cdots \sum_{q_1<q_2<\ldots<q_k} \chi_{q_1}\chi_{q_2}\cdots\chi_{q_k} + \cdots\right] \mathbb{P}_R \sum_{p=0}^{\infty} \frac{t^p}{p!} \mathcal{L}^p |A_1) \\
&= -\sum_{Z\neq\emptyset, Z\subset\{1,\cdots,n_*\}} (-1)^{|Z|} \prod_{q\in Z} \chi_q \cdot \mathbb{P}_R \sum_{p=0}^{\infty} \frac{t^p}{p!} \mathcal{L}^p |A_1).
\end{aligned} \tag{S26}$$

Proof. For each sequence in $\prod_{q=1}^{n_*}(1-\chi_q)\mathbb{P}_R \sum_{p=0}^{\infty} \frac{t^p}{p!}\mathcal{L}^p|A_1)$, if sequence $\mathcal{L}_{\beta_\ell}\cdots\mathcal{L}_{\beta_1}|A_1)$ is not creeping then it vanishes; if it is creeping then by Lemma 6 it vanishes. Hence $\mathbb{P}_R \prod_{q=1}^{n_*}(1-\chi_q)\sum_{p=0}^{\infty} \frac{t^p}{p!}\mathcal{L}^p|A_1) = 0$. In the second line of (S26) we simply expand the polynomial of χ_q, and in the last line we simply rewrite the result. □

To bound $\chi_q \mathrm{e}^{\mathcal{L}t}$, we now need to classify every term in $\chi_q \mathrm{e}^{Lt}$ by the *irreducible q-forward sequence* $\beta = (\beta_1, \ldots, \beta_\ell)$, constructed as follows: run the constructive algorithm of Lemma 5 to find the q-forward subsequence $\beta' \subseteq (\beta_1, \ldots, \beta_p)$, and then truncate the tail of β' such that $\ell(\beta') = N_q$. We denote the set of irreducible q-forward sequences $\mathcal{F}_q$. Sequences with the same irreducible q-forward sequence can be resummed as follows:

Lemma 8.

$$\chi_q \mathrm{e}^{\mathcal{L}t}|A_1) = \sum_{\beta\in\mathcal{F}_q} \int_{\Delta^\ell(t)} \mathrm{d}t_\ell \cdots \mathrm{d}t_1 \mathrm{e}^{\mathcal{L}(t-t_\ell)} \mathcal{L}_{\beta_\ell} \mathrm{e}^{\mathcal{L}^\beta_\ell (t_\ell - t_{\ell-1})} \mathcal{L}_{\beta_{\ell-1}} \mathrm{e}^{\mathcal{L}^\beta_{\ell-1}(t_{\ell-1}-t_{\ell-2})} \cdots \mathcal{L}_{\beta_1} \mathrm{e}^{\mathcal{L}^\beta_1 t_1} |A_1) \tag{S27}$$

where $\ell = \ell(\beta)$,

$$\mathcal{L}^\beta_p := \mathcal{L} - \sum_{\lambda\in Y^q_p(\beta)} \mathcal{L}_\lambda \tag{S28}$$

with

$$Y^q_p(\beta) := \{(q', k) : (k+1)2^{q'-1} \geq (k(\beta_p)+1)2^{q-1}\}, \tag{S29}$$

and $\Delta^\ell(t)$ denotes the ℓ-simplex:

$$\Delta^\ell(t) := \{(t_1, \ldots, t_\ell) \in [0,t]^\ell : t_1 \leq t_2 \leq \cdots \leq t_\ell\} \tag{S30}$$

with volume

$$\int_{\Delta^\ell(t)} \mathrm{d}t_\ell \cdots \mathrm{d}t_1 = \frac{t^\ell}{\ell!}. \tag{S31}$$

Proof. This is proved mirroring the proof of Theorem 4 of [16]. First, we show that

$$\chi_q \mathrm{e}^{\mathcal{L}t} = \sum_{\lambda\in\mathcal{F}_q} \sum_{m_0,\ldots,m_{\ell(\lambda)}=0}^{\infty} \frac{(t\mathcal{L})^{m_{\ell(\lambda)}}(t\mathcal{L}_{\lambda_{\ell(\lambda)}})(t\mathcal{L}^{\lambda}_{\ell(\lambda)})^{m_{\ell(\lambda)-1}}\cdots(t\mathcal{L}^{\lambda}_2)^{m_1}(t\mathcal{L}_{\lambda_1})(t\mathcal{L}^{\lambda}_1)^{m_0}}{(\ell(\lambda)+\sum_{j=0}^{\ell(\lambda)} m_j)!} \tag{S32}$$

with $\mathcal{L}^{\beta}_p$ defined in (S28). Every sequence on the right hand side of (S32) correspond to a term on the left because each of these sequences contains a $\lambda\in\mathcal{F}_q$ and thus has a long q-forward subsequence. Next, every sequence on the left can be written as a sequence on the right: by construction, the $Y^q_p(\beta)$ sets of couplings are chosen so that $\mathcal{L}^{\beta}_p$ does not change the irreducible q-forward subsequence of the term. The uniqueness of irreducible q-forward path implies that every term on the right hand side shows up exactly once. As the coefficients of terms on both sides of (S32) are the same, and we have found a bijection between the terms on both sides of the proposed equality (S32), we have demonstrated its veracity.

Secondly, we invoke a "generalized Schwinger-Karplus" identity proved in [16], which equates the right hand side of (S32) to the right hand side of (S27). □

$\chi_{q_1}\chi_{q_2}\cdots\chi_{q_k}\mathrm{e}^{\mathcal{L}t}$ can be understood by putting each χ_{q_i} together "indepedently." Indeed, we can classify every term in $\chi_{q_1}\chi_{q_2}\cdots\chi_{q_k}\mathrm{e}^{\mathcal{L}t}$ by the irreducible q-forward sequence at each scale q relatively independently: the only extra data we need is how the sequences weave between each other (i.e., the relative orders of all couplings between the long q-forward sequences for $q\in Z$). Defining $\mathcal{F}_Z$ as the set of sequences composed of the weaving together of $\beta^q\in\mathcal{F}_q, q\in Z$ (irreducible Z-forward sequences), we arrive at the following lemma:

Lemma 9.

$$\begin{aligned}\chi_{q_1}\chi_{q_2}\cdots\chi_{q_k}\mathrm{e}^{\mathcal{L}t}|A_1) &= \prod_{q\in Z}\chi_q\cdot\mathrm{e}^{\mathcal{L}t}|A_1)\\ &= \sum_{\beta\in\mathcal{F}_Z}\int_{\Delta^{\ell}(t)}\mathrm{d}t_\ell\cdots\mathrm{d}t_1\mathrm{e}^{\mathcal{L}(t-t_\ell)}\mathcal{L}_{\beta_\ell}\mathrm{e}^{\mathcal{L}^{\beta}_{\ell}(t_\ell-t_{\ell-1})}\mathcal{L}_{\beta_{\ell-1}}\mathrm{e}^{\mathcal{L}^{\beta}_{\ell-1}(t_{\ell-1}-t_{\ell-2})}\cdots\mathcal{L}_{\beta_1}\mathrm{e}^{\mathcal{L}^{\beta}_1 t_1}|A_1)\end{aligned} \tag{S33}$$

where $\ell=\ell(\beta)$,

$$\mathcal{L}^{\beta}_p := \mathcal{L} - \sum_{\lambda\in Y^Z_p(\beta)}\mathcal{L}_\lambda \tag{S34}$$

with

$$Y^q_p(\beta) := \{(q',k) : (k+1)2^{q'-1} \geq (k(\beta_p)+1)2^{q(\beta_p)-1}\}, \tag{S35}$$

Proof. The proof follows that of Lemma 8. First, we show that

$$\prod_{q\in Z}\chi_q\cdot\mathrm{e}^{\mathcal{L}t}|A_1) = \sum_{\lambda\in\mathcal{F}_Z}\sum_{m_0,\ldots,m_{\ell(\lambda)}=0}^{\infty}\frac{(t\mathcal{L})^{m_{\ell(\lambda)}}(t\mathcal{L}_{\lambda_{\ell(\lambda)}})(t\mathcal{L}^{\lambda}_{\ell(\lambda)})^{m_{\ell(\lambda)-1}}\cdots(t\mathcal{L}^{\lambda}_2)^{m_1}(t\mathcal{L}_{\lambda_1})(t\mathcal{L}^{\lambda}_1)^{m_0}}{(\ell(\lambda)+\sum_{j=0}^{\ell(\lambda)} m_j)!} \tag{S36}$$

with $\mathcal{L}^{\beta}_p$ defined in (S34). Every sequence on the right hand side of (S36) correspond to a term on the left because each of these sequences contains a $\lambda\in\mathcal{F}_Z$ as a subsequence and hence has a long q-forward subsequence for each $q\in Z$. Next, every sequence on the left can be written as a sequence on the right: by construction, the $Y^Z_p(\beta)$ sets of couplings are chosen so that $\mathcal{L}^{\beta}_p$ does not change the irreducible Z-forward subsequence of the term. The uniqueness of irreducible Z-forward subsequences also implies that every term on the right hand side shows up exactly once. As the coefficients of terms on both sides of (S36) are the same, and we have found a bijection between the terms on both sides of the proposed equality (S36), we have demonstrated its veracity.

Secondly, the generalized Schwinger-Karplus identity equates the right hand side of (S36) to the right hand side of (S33). □

The remainder of the proof is entirely combinatorial. As in (11), all quantum interference will now be hidden in the factors of $\mathrm{e}^{\mathcal{L}^{\lambda}_j t}$ in (S33). We begin with the following lemma:

Lemma 10.

$$\frac{\|\mathbb{P}_R e^{\mathcal{L}t}|A_1)\|}{2d\||A_1)\|} \leq -1 + \exp\left[\sum_{q=1}^{n_*} \binom{2^{1-q}R}{N_q} \frac{(2|t|)^{N_q}}{N_q!} \left(\sup_k \|\mathcal{L}_{(q,k)}\|\right)^{N_q}\right]. \tag{S37}$$

Proof. We begin by combining (S26) and (S33):

$$\|\mathbb{P}_R e^{\mathcal{L}t}|A_1)\| = \left\|\mathbb{P}_R \sum_Z (-1)^{|Z|} \sum_{\beta\in\mathcal{F}_Z} \int_{\Delta^\ell(t)} dt_\ell \cdots dt_1 e^{\mathcal{L}(t-t_\ell)} \mathcal{L}_{\beta_\ell} e^{\mathcal{L}_\ell^\beta (t_\ell - t_{\ell-1})} \cdots \mathcal{L}_{\beta_1} e^{\mathcal{L}_1^\beta t_1} |A_1)\right\| \tag{S38}$$

where $\ell := \ell(\beta)$. Since for all individual couplings, $\mathcal{L}_{mn}$ is an antisymmetric superoperator, each $\mathcal{L}_p^\beta$ is antisymmetric, and $e^{\mathcal{L}_p^\beta s}$ is orthogonal for any $s \in \mathbb{R}$. Using Lemma 4, we obtain

$$\begin{aligned}
\|\mathbb{P}_R e^{\mathcal{L}t}|A_1)\| &\leq 2d \left\|\sum_Z (-1)^{|Z|} \sum_{\beta\in\mathcal{F}_Z} \int_{\Delta^\ell(t)} dt_\ell \cdots dt_1 e^{\mathcal{L}(t-t_\ell)} \mathcal{L}_{\beta_\ell} e^{\mathcal{L}_\ell^\beta (t_\ell - t_{\ell-1})} \cdots \mathcal{L}_{\beta_1} e^{\mathcal{L}_1^\beta t_1} |A_1)\right\| \\
&\leq 2d \sum_Z \sum_{\beta\in\mathcal{F}_Z} \int_{\Delta^\ell(t)} dt_\ell \cdots dt_1 \left\|e^{\mathcal{L}(t-t_\ell)} \mathcal{L}_{\beta_\ell} e^{\mathcal{L}_\ell^\beta (t_\ell - t_{\ell-1})} \cdots \mathcal{L}_{\beta_1} e^{\mathcal{L}_1^\beta t_1} |A_1)\right\| \\
&\leq 2d \sum_Z \sum_{\beta\in\mathcal{F}_Z} \prod_{j=1}^{\ell} (\|\mathcal{L}_{\beta_j}\|) \cdot \||A_1)\| \int_{\Delta^\ell(t)} dt_\ell \cdots dt_1 \\
&\leq 2d\||A_1)\| \sum_Z \sum_{\beta\in\mathcal{F}_Z} \frac{(|t|)^\ell}{\ell!} \prod_{j=1}^{\ell} \sup_k \|\mathcal{L}_{(q(\beta_j),k)}\|
\end{aligned} \tag{S39}$$

where in the first line we used Proposition 3; in the second line we used the triangle inequality; in the third line we used the properties of Liouvillians described in Lemma 4 along with the fact that by construction each $\mathcal{L}_{(q,k)}$ in the irreducible sequence moves the operator to the right in such a way that we may use the effective norm from Lemma 4, and in the fourth line we computed the volume of the simplex $\Delta^\ell(t)$ as well as upper bounded $\|H_{\beta_j}\|$.

Next, we count the number of irreducible q-forward sequences, which is simply the number of possible ways to choose N_q different couplings out of $2^{1-q}R - 1$ different choices of k:

$$|\mathcal{F}_{\{q\}}| = \binom{2^{1-q}R}{N_q} \tag{S40}$$

To justify the factor of $2^{1-q}R - 1$, observe that the maximal value of k in $\mathcal{L}_{(q,k)}$ occurs when $(k+2)2^{q-1} = R$: $k \leq 2^{1-q}R - 2$. Since $k \geq 0$, we find $2^{1-q}R - 1$ different values of k.

The irreducible q-forward subsequences of any irreducible Z-forward sequence $\lambda \in \mathcal{F}_Z$ are completely independent of each other. Thus, the number of irreducible Z-forward sequences is given by product of the number of irreducible q-forward sequences for each $q \in Z$, together with the number of ways to weave together the few sequences:

$$|\mathcal{F}_Z| = \frac{(\sum_{q_1\in Z} N_{q_1})!}{\prod_{q_2\in Z} N_{q_2}!} \prod_{q\in Z} |\mathcal{F}_{\{q\}}|. \tag{S41}$$

Since if $\beta \in \mathcal{F}_Z$, $\ell(\beta) = \sum_{q\in Z} N_q$, we can combine (S39) and (S41) to obtain

$$\begin{aligned}
\frac{\|\mathbb{P}_R e^{\mathcal{L}t}|A_1)\|}{2d\||A_1)\|} &\leq \sum_Z \frac{(|t|)^{\sum_{q_1\in Z} N_{q_1}}}{(\sum_{q_1\in Z} N_{q_1})!} \frac{(\sum_{q_1\in Z} N_{q_1})!}{\prod_{q_2\in Z} N_{q_2}!} \prod_{q\in Z} \left(|\mathcal{F}_{\{q\}}| \left(\sup_k \|\mathcal{L}_{(q,k)}\|\right)^{N_q}\right) \\
&\leq \sum_Z \prod_{q\in Z} \left(|\mathcal{F}_{\{q\}}| \left(\sup_k \|\mathcal{L}_{(q,k)}\|\right)^{N_q} \frac{(|t|)^{N_q}}{N_q!}\right) \\
&\leq -1 + \prod_{q=1}^{n_*} \left[1 + |\mathcal{F}_{\{q\}}| \left(\sup_k \|\mathcal{L}_{(q,k)}\|\right)^{N_q} \frac{(|t|)^{N_q}}{N_q!}\right],
\end{aligned} \tag{S42}$$

where in the first two lines we made algebraic simplifications, and in the third line we used the distributive property together with the fact that there exist at least one scale with long q-forward sequence, i.e. $Z \in \mathbb{Z}_2^{\{1,\ldots,n_*\}} - \emptyset$. Combining (S40) with (S42) and the elementary identity $1 + x \leq \mathrm{e}^x$ for any $x \in \mathbb{R}$, we obtain (S37). $\square$

The last step proving of Theorem 1 is simplifying the sum in the exponential of (S37). Plugging Lemma 4 into (S37), we obtain

$$\begin{aligned} \frac{\|\mathbb{P}_R \mathrm{e}^{\mathcal{L}t}|A_1)\|}{2d\||A_1)\|} &\leq -1 + \exp\left[\sum_{q=1}^{n_*} \binom{2^{1-q}R - 1}{N_q} \frac{1}{N_q!} \left(\frac{2b|t|}{2^{q(\alpha'-1)}}\right)^{N_q}\right] \qquad \text{(S43)} \\ &\leq -1 + \exp\left[\sum_{q=1}^{n_*} \frac{(2^{1-q}R)^{N_q}}{N_q!^2} \left(\frac{2b|t|}{2^{q(\alpha'-1)}}\right)^{N_q}\right] \\ &\leq -1 + \exp\left[\sum_{q=1}^{n_*} \left(\frac{R}{2^q N_q^2} \frac{4\mathrm{e}^2 b|t|}{2^{q(\alpha'-1)}}\right)^{N_q}\right] \qquad \text{(S44)} \end{aligned}$$

where in the second line, we overestimated the choose function, and in the third line we used the inequality $n! > (n/\mathrm{e})^n$ for any $n \in \mathbb{N}$. It is useful to determine the first value q_* at which a long q-forward path has a single coupling: $N_q = 1$ for $q \geq q_*$. This occurs when

$$\frac{M}{R} \geq \frac{1}{2^{1+q_*\alpha'/2}}, \qquad \text{(S45)}$$

where we defined

$$M = \sum_{q=1}^{n_*} 2^{-q(\alpha'-2)/2}. \qquad \text{(S46)}$$

Then, combining (S20) and (S44), we obtain

$$\sum_{q=1}^{n_*} \left(\frac{R}{2^q N_q^2} \frac{4\mathrm{e}^2 b|t|}{2^{q(\alpha'-1)}}\right)^{N_q} < \sum_{q=1}^{q_*-1} \left(16\mathrm{e}^2 b|t| \frac{M^2}{R}\right)^{N_q} + 4\mathrm{e}^2 b|t| \sum_{q=q_*}^{n_*} \frac{R}{2^{q\alpha'}}. \qquad \text{(S47)}$$

We now analyze this sum for different ranges of α'.

Case 1: $\alpha' > 2$. In this regime, we begin by noting that

$$N_1 > N_2 > \cdots > N_{q_*-1}. \qquad \text{(S48)}$$

To derive this, note that the argument of the ceiling function in (S20) changes by a factor of $2^{\alpha'/2}$ each time q changes by 1. When $\alpha' > 2$, this factor is larger than 2, so once the argument is larger than 1, it changes by at least 1: $N_q \leq N_{q-1} - 1$. Hence we may write

$$\sum_{q=1}^{n_*} \left(\frac{R}{2^q N_q^2} \frac{4\mathrm{e}^2 b|t|}{2^{q(\alpha'-1)}}\right)^{N_q} < \sum_{n=1}^{\infty} \left(16\mathrm{e}^2 b|t| \frac{M^2}{R}\right)^{n} + 4\mathrm{e}^2 b|t| \sum_{q=q_*}^{n_*} \frac{R}{2^{q\alpha'}} \qquad \text{(S49)}$$

Next, we note that

$$M < \sum_{q=1}^{\infty} 2^{-q(\alpha'-2)/2} = \frac{1}{1 - 2^{-(\alpha'-2)/2}}. \qquad \text{(S50)}$$

which implies that

$$q_* \geq -1 + \frac{2}{\alpha'} \log_2 \frac{R}{M} = -1 + \frac{2n_*}{\alpha'} - \frac{2}{\alpha'} \log_2 \frac{1}{1 - 2^{-(\alpha'-2)/2}}. \qquad \text{(S51)}$$

We conclude that

$$\sum_{q=q_*}^{n_*} \frac{R}{2^{q\alpha'}} < \frac{2^{\alpha'}}{R(1 - 2^{-(\alpha'-2)/2})^2} \sum_{n=0}^{\infty} 2^{-\alpha' n} = \frac{2^{\alpha'}}{R(1 - 2^{-(\alpha'-2)/2})^2 (1 - 2^{-\alpha'})} \qquad \text{(S52)}$$

Case 2: $1 < \alpha' < 2$. In this regime, we must replace (S48) with the slightly weaker inequality

$$N_1 > N_3 > N_5 \cdots > N_{2\lceil q_*/2 \rceil - 1}, \tag{S53}$$

because the argument of (S20) now only varies by $2^{\alpha'/2} \geq \sqrt{2}$ each time q varies by 1. Moreover, we now find

$$M = \sum_{q=1}^{n_*} 2^{q(2-\alpha')/2} < R^{(2-\alpha')/2} \sum_{q'=0}^{\infty} 2^{-q'(2-\alpha')/2} = \frac{R^{(2-\alpha')/2}}{1 - 2^{-(2-\alpha')/2}} \tag{S54}$$

and that

$$q_* = -1 + \frac{2}{\alpha'} \log_2 \left(\left(1 - 2^{-(2-\alpha')/2}\right) R^{\alpha'/2} \right) = -1 + n_* - \frac{2}{\alpha'} \log_2 \frac{1}{1 - 2^{-(2-\alpha')/2}}. \tag{S55}$$

Hence, we obtain

$$\sum_{q=1}^{n_*} \left(\frac{R}{2^q N_q^2} \frac{4\mathrm{e}^2 b|t|}{2^{q(\alpha'-1)}} \right)^{N_q} < 2 \sum_{n=1}^{\infty} \left(16\mathrm{e}^2 b|t| \frac{M^2}{R} \right)^n + \frac{4\mathrm{e}^2 b|t|}{1 - 2^{-\alpha'}} \frac{2^{\alpha'}}{(1 - 2^{-(2-\alpha')/2})^2 R^{\alpha'-1}} \tag{S56}$$

where the 2 prefactor is a loose bound coming from that the $\sqrt{2}$ scaling - N_1 might equal to N_2.

Case 3: $\alpha' = 2$. In this regime, we obtain (S48),

$$M = n_*, \tag{S57}$$

and

$$N_q = \left\lceil \frac{1}{2} \frac{R}{2^q \log_2 R} \right\rceil, \tag{S58}$$

implying that

$$q_* \geq \log_2 \frac{R}{2 \log_2 R}. \tag{S59}$$

Hence we may write

$$\sum_{q=1}^{n_*} \left(\frac{R}{2^q N_q^2} \frac{4\mathrm{e}^2 b|t|}{2^{q(\alpha'-1)}} \right)^{N_q} < \sum_{n=1}^{\infty} \left(16\mathrm{e}^2 b|t| \frac{\log_2^2 R}{R} \right)^n + \frac{16}{3} \mathrm{e}^2 b|t| \frac{4 \log_2^2 R}{R} \tag{S60}$$

Each of the three cases leads to a simple bound. As a function of time, we obtain

$$\frac{\|\mathbb{P}_R \mathrm{e}^{\mathcal{L}t} |A_1)\|}{2\| |A_1)\|} \leq \frac{c_1 t}{\mathcal{R} - c_1 |t|} + c_2 \frac{|t|}{\mathcal{R}} \tag{S61}$$

where

$$\mathcal{R}(R) = \begin{cases} R & \alpha > 2 \\ R \log^{-2} R & \alpha = 2 \\ R^{\alpha-1} & 1 < \alpha < 2 \end{cases}, \tag{S62a}$$

$$c_1 = b \cdot \begin{cases} 16\mathrm{e}^2 (1 - 2^{-(\alpha-2)/2})^{-2} & \alpha > 2 \\ 16\mathrm{e}^2 & \alpha = 2 \\ 32\mathrm{e}^2 (1 - 2^{-(2-\alpha)/2})^{-2} & 1 < \alpha < 2 \end{cases}, \tag{S62b}$$

$$c_2 = b \cdot \begin{cases} 2^{2+\alpha} \mathrm{e}^2 (1 - 2^{-\alpha})^{-1} (1 - 2^{-(\alpha-2)/2})^{-2} & \alpha > 2 \\ \frac{64}{3} \mathrm{e}^2 & \alpha = 2 \\ 2^{2+\alpha} \mathrm{e}^2 (1 - 2^{-\alpha})^{-1} (1 - 2^{-(2-\alpha)/2})^{-2} & 1 < \alpha < 2 \end{cases}, \tag{S62c}$$

and $c_{1,2}$ are O(1) constant. For simplicity in these final two paragraphs, we will take the values of b calculated in frustrated models where $\alpha' = \alpha$. Analogous results hold for other models. Now observe that

$$\frac{\|\mathbb{P}_R \mathrm{e}^{\mathcal{L}t} |A_1)\|}{2d\| |A_1)\|} \leq (2c_1 + c_2) \frac{|t|}{\mathcal{R}}, \quad \left(|t| < \frac{\mathcal{R}}{2c_1} \right). \tag{S63}$$

Recall the definition of the scrambling time $t_{\mathrm{s}}^{\delta}(R)$ from (4). Using Proposition 2, we conclude that

$$\frac{\delta}{2} \leq 2d(2c_1 + c_2)\frac{t_{\mathrm{s}}^{\delta}(R)}{\mathcal{R}} \tag{S64}$$

Since $\delta < 2$, the right hand side becomes larger than 1 before the inequality (S63) breaks down. Since $\frac{1}{2}R \geq r \geq R$, we conclude that

$$t_{\mathrm{s}}^{\delta}(r) \geq \frac{\delta}{2db} \cdot \begin{cases} \dfrac{(1-2^{-\alpha})(1-2^{-(2-\alpha)/2})^2)}{(32+2^{2+\alpha})\mathrm{e}^2} r & \alpha > 2 \\ \dfrac{3}{160\mathrm{e}^2}\dfrac{r}{\log^2 r} & \alpha = 2 \\ \dfrac{(1-2^{-\alpha})(1-2^{-(2-\alpha)/2})^2)}{(64+2^{2+\alpha})\mathrm{e}^2} r^{\alpha-1} & 1 < \alpha < 2 \end{cases} . \tag{S65}$$

This proves the main Theorem 1. $\square$